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GENERAL SOLUTIONS FOR COUPLED EQUATIONS FOR PIEZOELECTRIC MEDIA

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Abstract—Three general solutions are obtained for the coupled dynamic equations for a transversely isotropic piezoelectric medium. These solutions are expressed in terms of the two functions ψ and F, where ψ satisfies a second-degree partial differential equation and F a sixth-degree partial differential equation, respectively. If the terms concerning the derivatives of time are removed, the results become three general solutions for the corresponding equilibrium equations, in which the function F can be represented by functions F_i $(i = 1, 2, 3)$, each of which satisfies a second-degree partial differential equation by utilizing a generalized Almansi theorem; and the solution Wang and Zheng [*Int. J. Solids Structures* **32,** 105–115 (1995)] obtained is proved to be consistent with one case of one of the three general solutions. When the constants $e_{15} = e_{31} = e_{33} = 0$ the piezo-electric coupling is absent; then, two of the solutions reduce to the elasticity general solutions for a transversely isotropic medium, one of which is the result Hu [*Acta Scientia Sin.* **2**(2), 145–151 (1953)] obtained; the other one has not been published. Last, the solution in the limiting explicit form for the problem for a half-space with concentrated loads at the boundary is obtained by utilizing the general solutions. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

According to a survey made by Rao and Sunar (1994), the control of flexible structures has attracted a considerable amount of research in recent years. The characteristic phenomena of piezo-electric materials, the direct and converse piezo-electric effects, permit them to be used as sensors and actuators in a control system. This motivated the investigations of these kinds of materials over the course of many years and many important achievements have been made in the subject such as fracture mechanics (Sosa, 1992), laminated structures (Sosa and Castro, 1993), numerical analysis (Sung Kyu Ha *et al.*, 1993) and structural control (Rao and Sunar, 1994). Wang and Zheng (1995) first carried out investigations about the general solution for three-dimensional problems for the piezo-electric media.

As suggested by Sosa and Castro (1993), the governing equations for the theory of piezo-electricity are:

$$\sigma_{ij,j} = -f_i + \rho \frac{\partial^2 u_i}{\partial t^2} \tag{1}$$

$$D_{j,j} = \rho_f \tag{2}$$

$$\sigma_{ij} = C_{ijkl}\bar{\varepsilon}_{kl} - e_{kij}E_k \tag{3}$$

$$D_i = e_{ikl}\bar{\varepsilon}_{kl} + \varepsilon_{ik}E_k \tag{4}$$

$$\bar{\varepsilon}_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{5}$$

$$E_i = -\phi_{,i} \tag{6}$$

where σ_{ij}, $\bar{\varepsilon}_{ij}$, u_i, E_i, and D_i are the components of stress, strain, displacement, electric field and electric displacement, respectively; ϕ is the electric potential; ρ, f_i, ρ_f are material density, body force, and density of free charges, respectively; and c_{ijkl}, e_{kij}, and ε_{ij} are the elastic stiffness, piezo-electric and dielectric constants, respectively. In the most general case of anisotropy, there are altogether 45 independent constants in eqns (3) and (4), which include 21 elastic stiffness, 18 piezo-electric and 6 dielectric constants. The present study is concerned, in particular, with the transversely isotropic piezo-electric media as they represent what is possibly the most technologically important piezo-electric material. Thus only 10 independent material constants are presented in eqns (3) and (4), which include 5 elastic stiffness, 3 piezo-electric and 2 dielectric constants.

If $e_{ijk} = 0$ the piezo-electric coupling is absent, eqns (1)–(6) reduce to 2 groups of equations for the uncoupled elastic and dielectric problems, respectively. One group consists of equations (1), (3) and (5), and the other one (2), (4) and (6).

The dynamic equations (1) and (2) can be expressed in terms of u_i and ϕ by virtue of eqns (3) and (6). In the context of transversely isotropic piezo-electric media and in the absence of f_i, ρ_f, they are

$$\left(c_{11}\frac{\partial^2}{\partial x^2}+c_{66}\frac{\partial^2}{\partial y^2}+c_{44}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2}\right)u+(c_{12}+c_{66})\frac{\partial^2 v}{\partial x\,\partial y}+(c_{13}+c_{44})\frac{\partial^2 w}{\partial x\,\partial z}+(e_{15}+e_{31})\frac{\partial^2\phi}{\partial x\,\partial z}=0 \quad (7)$$

$$(c_{12}+c_{66})\frac{\partial^2 u}{\partial x\,\partial y}+\left(c_{66}\frac{\partial^2}{\partial x^2}+c_{11}\frac{\partial^2}{\partial y^2}+c_{44}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2}\right)v+(c_{13}+c_{44})\frac{\partial^2 w}{\partial y\,\partial z}+(e_{15}+e_{31})\frac{\partial^2\phi}{\partial y\,\partial z}=0 \quad (8)$$

$$(c_{13}+c_{44})\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left(c_{44}\Lambda+c_{33}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2}\right)w+\left(e_{15}\Lambda+e_{33}\frac{\partial^2}{\partial z^2}\right)\phi=0 \quad (9)$$

$$(e_{15}+e_{31})\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left(e_{15}\Lambda+e_{33}\frac{\partial^2}{\partial z^2}\right)w-\left(\varepsilon_{11}\Lambda+\varepsilon_{33}\frac{\partial^2}{\partial z^2}\right)\phi=0, \quad (10)$$

where

$$\Lambda=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

Removing the inertia terms in the above equations, we have the corresponding equilibrium equations for which Wang and Zheng (1995) have given a general solution. This paper, however, concerned with the dynamic equations above, obtained three general solutions expressed in terms of two functions ψ and F where ψ satisfies a second-degree partial differential equation and F a sixth-degree partial differential equation. If all the functions in those solutions are independent of time, these results shall become three general solutions for the equilibrium equations. At this stage, each general solution possesses three cases individually in accordance with the possibilities that the eigenvalues s_i^2, (i = 1, 2, 3) might be equal to each other. Included is the proof of the solution Wang and Zheng (1995) obtained for only one case of the three general solutions and does not include the other two cases of that specific solution. This indicates the extensiveness of the general solutions presented in this paper.

2. GENERAL SOLUTIONS OF DYNAMIC EQUATIONS

Firstly, introduce displacement functions ψ and G to represent the components of displacement u and v, giving

$$u = \frac{\partial \psi}{\partial y} - \frac{\partial G}{\partial x}, \quad v = -\frac{\partial \psi}{\partial x} - \frac{\partial G}{\partial y}. \tag{11}$$

Substituting eqn (11) into eqns (7) and (8), we have:

$$\frac{\partial B}{\partial y} - \frac{\partial A}{\partial x} = 0, \tag{12}$$

$$\frac{\partial B}{\partial x} + \frac{\partial A}{\partial y} = 0, \tag{13}$$

where

$$B = \left(c_{66}\Lambda + c_{44}\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2}\right)\psi \tag{14}$$

$$A = \left(c_{11}\Lambda + c_{44}\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2}\right)G - (c_{13}+c_{44})\frac{\partial w}{\partial z} - (e_{15}+e_{31})\frac{\partial \phi}{\partial z}. \tag{15}$$

The solution of eqns (12) and (13) can be found of the form:

$$A = \frac{\partial H}{\partial y}, \quad B = \frac{\partial H}{\partial x} \tag{16}$$

and H should satisfy

$$\Lambda H = 0. \tag{17}$$

It can be proved that eqn (16) can be simplified as follows (see Appendix A):

$$A = 0, \quad B = 0. \tag{18}$$

Substituting eqn (11) into eqns (9) and (10), then listing them together with eqn (18), we have

$$\left(c_{66}\Lambda + c_{44}\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2}\right)\psi = 0 \tag{19}$$

and

$$\mathbf{D}\begin{Bmatrix} G \\ w \\ \phi \end{Bmatrix} = 0, \tag{20}$$

where **D** is a operator's matrix:

$$\mathbf{D}=\begin{bmatrix} c_{11}\Lambda+c_{44}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2} & -(c_{13}+c_{44})\frac{\partial}{\partial z} & -(e_{15}+e_{31})\frac{\partial}{\partial z} \\ -(c_{13}+c_{44})\Lambda\frac{\partial}{\partial z} & c_{44}\Lambda+c_{33}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2} & e_{15}\Lambda+e_{33}\frac{\partial^2}{\partial z^2} \\ (e_{15}+e_{31})\Lambda\frac{\partial}{\partial z} & -\left(e_{15}\Lambda+e_{33}\frac{\partial^2}{\partial z^2}\right) & \varepsilon_{11}\Lambda+\varepsilon_{33}\frac{\partial^2}{\partial z^2} \end{bmatrix} \tag{21}$$

Calculate the determinant of matrix **D**, then introduce a new function F to let $|\mathbf{D}|F=0$, giving

$$\left[a\frac{\partial^6}{\partial z^6}+b\Lambda\frac{\partial^4}{\partial z^4}+c\Lambda\Lambda\frac{\partial^2}{\partial z^2}+d\Lambda\Lambda\Lambda+g\Lambda\frac{\partial^4}{\partial t^4}+h\Lambda\Lambda\frac{\partial^2}{\partial t^2} +k\Lambda\frac{\partial^2}{\partial z^2}\frac{\partial^2}{\partial t^2}+l\frac{\partial^4}{\partial z^4}\frac{\partial^2}{\partial t^2}+n\frac{\partial^2}{\partial z^2}\frac{\partial^4}{\partial t^4}\right]F=0, \tag{22}$$

where

$$\begin{aligned}
a &= c_{44}(e_{33}^2+c_{33}\varepsilon_{33})\\
b &= c_{33}[c_{44}\varepsilon_{11}+(e_{15}+e_{31})^2]+\varepsilon_{33}[c_{11}c_{33}+c_{44}^2-(c_{13}+c_{44})^2]\\
&\quad +e_{33}[2c_{44}e_{15}+c_{11}e_{33}-2(c_{13}+c_{44})(e_{15}+e_{31})]\\
c &= c_{44}[c_{11}\varepsilon_{33}+(e_{15}+e_{31})^2]+\varepsilon_{11}[c_{11}c_{33}+c_{44}^2-(c_{13}+c_{44})^2]\\
&\quad +e_{15}[2c_{11}e_{33}+c_{44}e_{15}-2(c_{13}+c_{44})(e_{15}+e_{31})]\\
d &= c_{11}(e_{15}^2+c_{44}\varepsilon_{11})\\
g &= \rho^2\varepsilon_{11}\\
h &= -\rho[e_{15}^2+(c_{11}+c_{44})\varepsilon_{11}]\\
k &= -\rho[2e_{15}e_{33}+(c_{44}+c_{33})\varepsilon_{11}+(c_{11}+c_{44})\varepsilon_{33}+(e_{15}+e_{31})^2]\\
l &= -\rho[e_{33}^2+(c_{44}+c_{33})\varepsilon_{33}]\\
n &= \rho^2\varepsilon_{33}.
\end{aligned}$$

Calculate the algebraic complements A_{ij} of $|\mathbf{D}|$ based on each row and let

$$G=A_{i1}F,\quad w=A_{i2}F,\quad \phi=A_{i3}F,\quad (i=1,2,3). \tag{23}$$

Then eqn (23) represents three solutions for eqn (20) if F satisfies eqn (22). We thus obtain three general solutions for the dynamic equations by substituting eqn (23) into eqn (11), yielding

$$\left.\begin{aligned}
u &= \frac{\partial\psi}{\partial y}-A_{i1}\frac{\partial F}{\partial x}\\
v &= -\frac{\partial\psi}{\partial x}-A_{i1}\frac{\partial F}{\partial y}\\
w &= A_{i2}F\\
\phi &= A_{i3}F\quad (i=1,2,3)
\end{aligned}\right\} \tag{24}$$

where ψ and F satisfy eqns (19) and (22), respectively, and

$$\left.\begin{aligned} A_{11} &= \left(c_{44}\Lambda+c_{33}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2}\right)\left(\varepsilon_{11}\Lambda+\varepsilon_{33}\frac{\partial^2}{\partial z^2}\right)+\left(e_{15}\Lambda+e_{33}\frac{\partial^2}{\partial z^2}\right)^2, \\ A_{12} &= \left[(c_{13}+c_{44})\left(\varepsilon_{11}\Lambda+\varepsilon_{33}\frac{\partial^2}{\partial z^2}\right)+(e_{15}+e_{31})\left(e_{15}\Lambda+e_{33}\frac{\partial^2}{\partial z^2}\right)\right]\Lambda\frac{\partial}{\partial z}, \\ A_{13} &= \left[(c_{13}+c_{44})\left(e_{15}\Lambda+e_{33}\frac{\partial^2}{\partial z^2}\right)-(e_{15}+e_{31})\left(c_{44}\Lambda+c_{33}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2}\right)\right]\Lambda\frac{\partial}{\partial z}, \end{aligned}\right\} \quad (25)$$

$$\left.\begin{aligned} A_{21} &= \left[(c_{13}+c_{44})\left(\varepsilon_{11}\Lambda+\varepsilon_{33}\frac{\partial^2}{\partial z^2}\right)+(e_{15}+e_{31})\left(e_{15}\Lambda+e_{33}\frac{\partial^2}{\partial z^2}\right)\right]\frac{\partial}{\partial z}, \\ A_{22} &= \left(c_{11}\Lambda+c_{44}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2}\right)\left(\varepsilon_{11}\Lambda+\varepsilon_{33}\frac{\partial^2}{\partial z^2}\right)+(e_{15}+e_{31})^2\Lambda\frac{\partial^2}{\partial z^2}, \\ A_{23} &= \left(c_{11}\Lambda+c_{44}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2}\right)\left(e_{15}\Lambda+e_{33}\frac{\partial^2}{\partial z^2}\right)-(c_{13}+c_{44})(e_{15}+e_{31})\Lambda\frac{\partial^2}{\partial z^2}, \end{aligned}\right\} \quad (26)$$

$$\left.\begin{aligned} A_{31} &= \left[(e_{15}+e_{31})\left(c_{44}\Lambda+c_{33}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2}\right)-(c_{13}+c_{44})\left(e_{15}\Lambda+e_{33}\frac{\partial^2}{\partial z^2}\right)\right]\frac{\partial}{\partial z}, \\ A_{32} &= (c_{13}+c_{44})(e_{15}+e_{31})\Lambda\frac{\partial^2}{\partial z^2}-\left(c_{11}\Lambda+c_{44}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2}\right)\left(e_{15}\Lambda+e_{33}\frac{\partial^2}{\partial z^2}\right), \\ A_{33} &= \left(c_{11}\Lambda+c_{44}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2}\right)\left(c_{44}\Lambda+c_{33}\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2}\right)-(c_{13}+c_{44})^2\Lambda\frac{\partial^2}{\partial z^2}. \end{aligned}\right\} \quad (27)$$

In cylindrical coordinates (r, θ, z), the general solutions (24) take the form

$$\left.\begin{aligned} u_r &= \frac{\partial\psi}{r\,\partial\theta}-\frac{\partial}{\partial r}A_{i1}F \\ u_\theta &= -\frac{\partial\psi}{\partial r}-\frac{\partial}{r\,\partial\theta}A_{i1}F \\ w &= A_{i2}F \\ \phi &= A_{i3}F\,(i=1,2,3) \end{aligned}\right\} \quad (28)$$

and in eqns (19), (22) and (25)–(27)

$$\Lambda = \frac{\partial^2}{\partial r^2}+\frac{\partial}{r\,\partial r}+\frac{\partial^2}{r^2\,\partial\theta^2},$$

correspondingly.

Furthermore, we shall obtain the general solutions for axisymmetric problems if we let $\psi = 0$ and F is independent of θ in eqn (28) or (24).

If we substitute eqn (24) into eqns (5) and (6), and then substitute the results into eqns (3) and (4), we can obtain representations of σ_{ij} and D_i. That means specific boundary conditions and initial conditions can also be expressed in terms of ψ, F and their partial derivatives.

3. GENERAL SOLUTIONS FOR EQUILIBRIUM EQUATIONS

Equation (24) becomes three general solutions of the equilibrium equations if all the functions in them are independent of time. Firstly, we get the equation which ψ satisfies at this time from eqn (19) as follows:

$$\left(\Lambda+\frac{\partial^2}{\partial z_0^2}\right)\psi = 0, \tag{29}$$

where $z_0^2 = s_0^2 z^2$, and

$$s_0^2 = \frac{c_{66}}{c_{44}}. \tag{30}$$

At the same time, eqn (22), which F satisfies, can be simplified to take the form

$$\left(\Lambda+\frac{\partial^2}{\partial z_1^2}\right)\left(\Lambda+\frac{\partial^2}{\partial z_2^2}\right)\left(\Lambda+\frac{\partial^2}{\partial z_3^2}\right)F = 0, \tag{31}$$

where $z_i^2 = s_i^2 z^2$, $(i = 1, 2, 3)$ and s_i^2 are the three roots of the equation

$$as^6 - bs^4 + cs^2 - d = 0. \tag{32}$$

The three roots s_i^2 can be expressed in terms of a, b, c, d and there will always exist a real one among them, assuming s_1^2 is the real root with no loss of generality. Besides, we further assume $\mathrm{Re}\,(s_i) > 0$ $(i = 1, 2, 3)$.

Correspondingly, eqns (25)–(27) can also be simplified. For example, eqn (26) may be rewritten as

$$\begin{aligned} A_{21} &= \left(m_1\Lambda + m_2\frac{\partial^2}{\partial z^2}\right)\frac{\partial}{\partial z} \\ A_{22} &= c_{11}\varepsilon_{11}\Lambda^2 + m_3\Lambda\frac{\partial^2}{\partial z^2} + c_{44}\varepsilon_{33}\frac{\partial^4}{\partial z^4} \\ A_{23} &= c_{11}e_{15}\Lambda^2 + m_4\Lambda\frac{\partial^2}{\partial z^2} + c_{44}e_{33}\frac{\partial^4}{\partial z^4}, \end{aligned} \tag{33}$$

where

$$\begin{aligned} m_1 &= \varepsilon_{11}(c_{13}+c_{44}) + e_{15}(e_{15}+e_{31}) \\ m_2 &= \varepsilon_{33}(c_{13}+c_{44}) + e_{33}(e_{15}+e_{31}) \\ m_3 &= c_{11}\varepsilon_{33} + c_{44}\varepsilon_{11} + (e_{15}+e_{31})^2 \\ m_4 &= c_{11}e_{33} + c_{44}e_{15} - (c_{13}+c_{44})(e_{15}+e_{31}). \end{aligned} \tag{34}$$

It is proved that there still exists a generalized Almansi's theorem in this case (see Appendix B), which is analogous to the works done by Eubanks and Sternberg (1954). The theorem is:

Let R be a region of the (x, y, z)-space such that a straight line parallel to the z-axis intersects the boundary of R at no more than two points, then in the region R, the solutions of eqn (31) can be represented as follows:

$$\begin{aligned}
&(1)\quad F = F_1+F_2+F_3 \quad \text{for } s_i^2\,(i=1,2,3) \quad \text{are distinct;}\\
&(2)\quad F = F_1+F_2+zF_3 \quad \text{for } s_1^2 \neq s_2^2 = s_3^2; \quad \text{and}\\
&(3)\quad F = F_1+zF_2+z^2F_3 \quad \text{for } s_1^2 = s_2^2 = s_3^2,
\end{aligned} \tag{35}$$

where F_i satisfy the following equations, respectively, giving

$$\left(\Lambda+\frac{\partial^2}{\partial z_i^2}\right)F_i = 0, \quad (i=1,2,3). \tag{36}$$

Hence, by utilizing this theorem, every solution of the six-degree partial differential equation (31) can be represented by the solutions of eqn (36) which is only second degree. Furthermore, the theorem can be used to rewrite our general solutions. For example, in the first case of eqn (35), by virtue of eqns (24), (33), (34) and (36), the general solution can be rewritten as

$$\left.\begin{aligned}
u &= \frac{\partial\psi}{\partial y}+\sum_{i=1}^{3}\alpha_i s_i\frac{\partial^4 F_i}{\partial x\,\partial z_i^3}\\
v &= -\frac{\partial\psi}{\partial x}+\sum_{i=1}^{3}\alpha_i s_i\frac{\partial^4 F_i}{\partial y\,\partial z_i^3}\\
w &= \sum_{i=1}^{3}\beta_i\frac{\partial^4 F_i}{\partial z_i^4}\\
\phi &= \sum_{i=1}^{3}\gamma_i\frac{\partial^4 F_i}{\partial z_i^4},
\end{aligned}\right\} \tag{37}$$

where

$$\begin{aligned}
\alpha_i &= m_1-m_2 s_i^2,\\
\beta_i &= c_{11}\varepsilon_{11}-m_3 s_i^2+c_{44}\varepsilon_{33}s_i^4,\\
\gamma_i &= c_{11}e_{15}-m_4 s_i^2+c_{44}e_{33}s_i^4.
\end{aligned} \tag{38}$$

For further simplification, let

$$\alpha_i s_i\frac{\partial^3 F_i}{\partial z_i^3} = \psi_i, \tag{39}$$

then eqn (37) becomes:

$$\left.\begin{aligned}
u &= -\frac{\partial\psi_0}{\partial y}+\sum_{i=1}^{3}\frac{\partial\psi_i}{\partial x}\\
v &= \frac{\partial\psi_0}{\partial x}+\sum_{i=1}^{3}\frac{\partial\psi_i}{\partial y}\\
w &= \sum_{i=1}^{3}k_{1i}\frac{\partial\psi_i}{\partial z}\\
\phi &= \sum_{i=1}^{3}k_{2i}\frac{\partial\psi_i}{\partial z},
\end{aligned}\right\} \tag{40}$$

where $\psi_0 = -\psi$ and

$$k_{1i} = \frac{\beta_i}{\alpha_i s_i^2}, \quad k_{2i} = \frac{\gamma_i}{\alpha_i s_i^2}. \tag{41}$$

Besides, in view of eqns (29), (36), and (39), we obtain the equation which ψ_i, respectively, satisfies:

$$\left(\Lambda + \frac{\partial^2}{\partial z_i^2}\right)\psi_i = 0, \quad (i = 0, 1, 2, 3). \tag{42}$$

Clearly, the general solution represented by eqn (40) is consistent with the result Wang and Zheng (1995) obtained.

4. GENERAL SOLUTIONS FOR DECOUPLED PROBLEMS

As stated before, when $e_{15} = e_{31} = e_{33} = 0$ piezo-electric coupling is absent, the governing equations reduce to two groups of equations which will be treated separately. At this stage, the general solutions represented by eqn (24) will also reduce correspondingly to the general solutions for the decoupled problems.

Firstly, in this case, eqn (22) can be rewritten as:

$$LMF = 0, \tag{43}$$

where L and M are two operators:

$$L = \left(c_{11}\Lambda + c_{44}\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2}\right)\left(c_{44}\Lambda + c_{33}\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2}\right) - (c_{13}+c_{44})^2\Lambda\frac{\partial^2}{\partial z^2}$$

$$M = \varepsilon_{11}\Lambda + \varepsilon_{33}\frac{\partial^2}{\partial z^2}. \tag{44}$$

Secondly, let

$$MF = f. \tag{45}$$

Then by virtue of eqn (43) we obtain the equation which f satisfies:

$$Lf = 0. \tag{46}$$

The three general solutions stated by eqn (24), corresponding to $i = 1, 2, 3$, reduce to the following three solutions, respectively:

$$\left.\begin{aligned} u &= \frac{\partial\psi}{\partial y} - \left(c_{44}\Lambda + c_{33}\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2}\right)\frac{\partial f}{\partial x} \\ v &= -\frac{\partial\psi}{\partial x} - \left(c_{44}\Lambda + c_{33}\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2}\right)\frac{\partial f}{\partial y} \\ w &= (c_{13}+c_{44})\Lambda\frac{\partial f}{\partial z} \\ \phi &= 0; \end{aligned}\right\} \tag{47}$$

$$\left.\begin{aligned} u &= \frac{\partial\psi}{\partial y} - (c_{13}+c_{44})\frac{\partial^2 f}{\partial x\,\partial z} \\ v &= -\frac{\partial\psi}{\partial x} - (c_{13}+c_{44})\frac{\partial^2 f}{\partial y\,\partial z} \\ w &= \left(c_{11}\Lambda + c_{44}\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2}\right) f \\ \phi &= 0; \end{aligned}\right\} \tag{48}$$

and

$$u = v = w = 0, \quad \phi = LF. \tag{49}$$

Equations (47) and (48) are the general elasticity solutions for transverse isotropy. For equilibrium problems, let $\varphi = (c_{13}+c_{44})f$, then eqn (48) is the general solution Hu (1953) obtained [see also Lekhniskii (1981)]. The general solution represented by eqn (47) has not been published. In view of eqn (43), the function ϕ in the third general solution stated by eqn (49) satisfies $M\phi = 0$ which is the equation the electric potential must satisfy in dielectric problems.

5. PROBLEM FOR A HALF-SPACE WITH LOADS AT THE BOUNDARY

Wang and Zheng (1995) investigated the problem of a piezo-electric half-space with concentrated loads at the boundary. Sosa and Castro's study (1994), however, was concerned with concentrated loads at the boundary of a piezo-electric half-plane. In present study, as an example of applications of the general solutions, we now consider the problem for a half-space, given the forces and electrical displacement at the boundary.

We assume the following boundary conditions on the surface of a half-space ($z \geqslant 0$), giving

$$\sigma_z = P_0(x,y), \quad \tau_{xz} = Q_0(x,y), \quad \tau_{yz} = 0, \quad D_z = T_0(x,y) \quad \text{on } z = 0. \tag{50}$$

The Fourier transform for a function $f(x,y)$ is defined as

$$\bar{f}(\alpha,\beta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,e^{i(\alpha x+\beta y)}\,dx\,dy. \tag{51}$$

Then, we can obtain the Fourier transforms of eqns (24), (33), (29), (31), and (50), respectively, yielding

$$\begin{aligned} \bar{u} &= -i\beta\bar{\psi} + i\alpha\bar{A}_{21}\bar{F} \\ \bar{v} &= i\alpha\psi + i\beta\bar{A}_{21}\bar{F} \\ \bar{w} &= \bar{A}_{22}\bar{F} \\ \bar{\phi} &= \bar{A}_{23}\bar{F}; \end{aligned} \tag{52}$$

$$\begin{aligned} \bar{A}_{21} &= m_2\frac{\partial^2}{\partial z^3} - m_1\rho^2\frac{\partial}{\partial z} \\ \bar{A}_{22} &= c_{44}\varepsilon_{33}\frac{\partial^4}{\partial z^4} - m_3\rho^2\frac{\partial^2}{\partial z^4} + c_{11}\varepsilon_{11}\rho^4 \\ \bar{A}_{23} &= c_{44}e_{33}\frac{\partial^4}{\partial z^4} - m_4\rho^2\frac{\partial^2}{\partial z^2} + c_{11}e_{15}\rho^4; \end{aligned} \tag{53}$$

$$\left(\frac{\partial^2}{\partial z_0^2}-\rho^2\right)\bar{\psi}=0; \tag{54}$$

$$\left(\frac{\partial^2}{\partial z_1^2}-\rho^2\right)\left(\frac{\partial^2}{\partial z_2^2}-\rho^2\right)\left(\frac{\partial^2}{\partial z_3^2}-\rho^2\right)\bar{F}=0 \tag{55}$$

and

$$\bar{\sigma}_z=\bar{P}_0,\quad \bar{\tau}_{xz}=\bar{Q}_0$$
$$\bar{\tau}_{yz}=0,\quad \bar{D}_z=\bar{T}_0\quad \text{on}\, z=0, \tag{56}$$

where

$$\rho^2=\alpha^2+\beta^2. \tag{57}$$

It has assumed that the function ψ and its first-degree partial derivative, F, and its partial derivatives of degree one to degree five tend to zero as $x^2+y^2\rightarrow\infty$.

The general solutions for eqns (54) and (55) can be readily obtained; under the conditions that the quantities u, v, w and ϕ tend to zero as $z\rightarrow\infty$, the functions $\bar{\psi}$ and $\bar{F}$ have the following form

$$\bar{\psi}=k_0\,\mathrm{e}^{-\rho s_0 z}; \tag{58}$$

$$\bar{F}=\sum_{n=1}^{3}k_n\,\mathrm{e}^{-\rho s_n z}\quad \text{for}\, s_i (i=1,2,3)\, \text{are distinct}; \tag{59a}$$

$$\bar{F}=k_1\,\mathrm{e}^{-\rho s_1 z}+k_2\,\mathrm{e}^{-\rho s_2 z}+k_3 z\,\mathrm{e}^{-\rho s_2 z}\quad \text{for}\, s_1\neq s_2=s_3; \tag{59b}$$

$$\bar{F}=k_1\,\mathrm{e}^{-\rho s_1 z}+k_2 z\,\mathrm{e}^{-\rho s_1 z}+k_3 z^2\,\mathrm{e}^{-\rho s_1 z}\quad \text{for}\, s_1=s_2=s_3, \tag{59c}$$

where $k_n (n=0,1,2,3)$ are arbitrary constants.

Substituting eqns (58) and (59) into eqn (52), we obtain the representations of $\bar{u}$, $\bar{v}$, $\bar{w}$ and $\bar{\phi}$. Then, by utilizing the Fourier transforms of the constitutive relations, the representations of the $\bar{\sigma}_z$, $\bar{\tau}_{yz}$, $\bar{\tau}_{xz}$, $\bar{D}$, etc. can be obtained. For example:

$$\bar{\sigma}_z=-\mathrm{i}c_{13}(\alpha\bar{u}+\beta\bar{v})+c_{33}\frac{\partial\bar{w}}{\partial z}+e_{33}\frac{\partial\bar{\phi}}{\partial z} \tag{60}$$

$$\bar{\tau}_{yz}=c_{44}\left(\frac{\partial\bar{v}}{\partial z}-\mathrm{i}\beta\bar{w}\right)-\mathrm{i}\beta e_{15}\bar{\phi} \tag{61}$$

$$\bar{\tau}_{xz}=c_{44}\left(\frac{\partial\bar{u}}{\partial z}-\mathrm{i}\alpha\bar{w}\right)-\mathrm{i}\alpha e_{15}\bar{\phi} \tag{62}$$

$$\bar{D}_z=-\mathrm{i}\alpha e_{31}\bar{u}-\mathrm{i}\beta e_{31}\bar{v}+e_{33}\frac{\partial\bar{w}}{\partial z}-\varepsilon_{33}\frac{\partial\bar{\phi}}{\partial z} \tag{63}$$

Last, by virtue of eqns (60)–(63) and (56), we can obtain four algebraic equations for k_n $(n=0,1,2,3)$. Solving this group of equations and substituting the values of k_n $(n=0,1,2,3)$ into eqn (52), then inverting the equation by applying inverse formula for the Fourier transform, we thus obtain the solutions for the boundary-value problem considered.

In particular, assuming that

$$P_0(x,y) = P\delta(x)\delta(y),$$
$$Q_0(x,y) = Q\delta(x)\delta(y),$$
$$T_0(x,y) = T\delta(x)\delta(y), \tag{64}$$

and taking the case of eqn (59a) as an example, we now can obtain the solutions for the boundary-value problem considered when s_1, s_2 and s_3 are distinct, giving

$$u = \frac{Qn_{50}}{d_x}\left(\frac{1}{R_0^*} - \frac{y^2}{R_0^{*2}R_0}\right) + \sum_{i=1}^{3}\left[\frac{Qn_{5i}}{d_x}\left(\frac{1}{R_i^*} - \frac{x^2}{R_i^{*2}R_i}\right) - \frac{Pl_{5i}x}{d_zR_i^*R_i} - \frac{Tb_{5i}x}{d_dR_i^*R_i}\right] \tag{65}$$

$$v = -\frac{Qn_{60}xy}{d_xR_0^{*2}R_0} - \sum_{i=1}^{3}\left(\frac{Qn_{6i}xy}{d_xR_i^{*2}R_i} + \frac{Pl_{6i}y}{d_zR_i^*R_i} + \frac{Tb_{6i}y}{d_dR_i^*R_i}\right) \tag{66}$$

$$w = \sum_{i=1}^{3}\left(\frac{Qn_{7i}x}{d_xR_i^*R_i} + \frac{Pl_{7i}}{d_zR_i} + \frac{Tb_{7i}}{d_dR_i}\right) \tag{67}$$

$$\phi = \sum_{i=1}^{3}\left(\frac{Qn_{8i}x}{d_xR_i^*R_i} + \frac{Pl_{8i}}{d_zR_i} + \frac{Tb_{8i}}{d_dR_i}\right), \tag{68}$$

where $R_i = \sqrt{x^2+y^2+z_i^2}$, $R_i^* = R_i + z_i$, and d_x, d_z, d_d, n_{ij}, l_{ij}, b_{ij} ($i = 5, 6, 7, 8; j = 0, 1, 2, 3$), which are tabulated in Appendix C, are coefficients expressed in terms of material constants.

6. CONCLUSIONS

(1) Three general solutions for the coupled dynamic equations for transversely-isotropic piezo-electric media have been obtained and are represented by eqn (24). They become the general solutions for the corresponding equilibrium equations on the removal of those terms concerning the derivatives of time.

(2) It is proved that the solution Wang and Zheng (1995) obtained is consistent with eqn (40), one case of one of the three general solutions obtained in this paper and does not include the other two cases of that specific solution.

(3) Two general elasticity solutions for the dynamic equations for transverse isotropy have been obtained and are represented by eqns (47) and (48). They become two general solutions for the equilibrium equations if all the functions in them are independent of time, one of which is the solution Hu (1953) obtained, the other one has not been published.

(4) For the problem of a half-space with concentrated loads on the surface, we can obtain limiting explicit solutions such as eqns (65)–(68) for the three cases of eqn (59) respectively, by utilizing the general solutions presented in the paper.

REFERENCES

Eubanks, R. A. and Sternberg, E. (1954). On the axisymmetric problem of elastic theory for a medium with transverse isotropy. *J. Ration. Mech. Anal.* **3**, 89–101.

Hu, H. (1953). On the three-dimensional problems of the theory of elasticity of a transversely isotropic body. *Acta Scientia Sin.* **2**(2), 145–151.

Lekhniskii, S. G. (1981). *Theory of Elasticity of an Anisotropic Body*. Mir Publishers, Moscow.

Rao, S. S. and Sunar, M. (1994). Piezoelectricity and it's use in disturbance sensing and control of flexible structures: a survey. *Appl. Mech. Rev.* **47**(4), 113–123.

Sosa, H. (1992). On the fracture mechanics of piezoelectric solid. *Int. J. Solid Structures* **29**(21), 2613–2622.

Sosa, H. A. and Castro, M. (1993). Electroelastic analysis of piezoelectric laminated structures. *Appl. Mech. Rev.* **46**(11), s21–s28.

Sosa, H. A. and Castro, M. A. (1994). On concentrated loads at the boundary of a piezoelectric half-plane. *J. Mech. Phys. Solids* **42**(7), 1105–1122.

Sung Kyu Ha, Keilers, C. and Fu-Kuo, C. (1993). Finite element analysis of composite structures containing distributed piezoceramic sensors and actuators. *AIAA J.* **30**(3), 772–780.

Wang, Z. and Zheng, B. (1995). The general solution of three-dimensional problems in piezoelectric media. *Int. J. Solids Structures* **32**(1), 105–115.

APPENDIX A

In this section we prove the possibility of eqn (18). If we see eqn (11) as the equation for the solutions of ψ and G by giving u and v, the resolvent form for displacement represented by eqn (11) is not single. In fact, the homogeneous equations of eqn (11) have nontrivial solutions namely ψ_0 and G_0 which satisfy:

$$\frac{\partial \psi_0}{\partial y} - \frac{\partial G_0}{\partial x} = 0, \quad \frac{\partial \psi_0}{\partial x} + \frac{\partial G_0}{\partial y} = 0. \tag{A1}$$

In view of eqn (A1), we have:

$$\Lambda \psi_0 = 0, \quad \Lambda G_0 = 0. \tag{A2}$$

Thus, if the functions ψ and G in eqn (11) are replaced by $\psi + \psi_0$ and $G + G_0$, respectively, they still represent the same displacement as before. Now, replace ψ and G by $\psi + \psi_0$ and $G + G_0$, respectively, in eqns (14) and (15), and then substitute the results into eqn (16). At this stage, if there exist:

$$\begin{aligned} \left(c_{11}\Lambda + c_{44}\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2}\right)G_0 &= \frac{\partial H}{\partial y} \\ \left(c_{66}\Lambda + c_{44}\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2}\right)\psi_0 &= \frac{\partial H}{\partial x}, \end{aligned} \tag{A3}$$

eqn (16) can be simplified into eqn (18). In view of eqn (A2), eqn (A3) can be rewritten as:

$$\begin{aligned} \left(c_{44}\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2}\right)G_0 &= \frac{\partial H}{\partial y} \\ \left(c_{44}\frac{\partial^2}{\partial z^2} - \rho\frac{\partial^2}{\partial t^2}\right)\psi_0 &= \frac{\partial H}{\partial x} \end{aligned} \tag{A4}$$

Introducing a coordinate transformation defined as

$$\xi = z - c_2 t, \quad \eta = z + c_2 t, \tag{A5}$$

where

$$c_2^2 = \frac{c_{44}}{\rho}, \tag{A6}$$

eqn (A4) becomes

$$\frac{\partial^2 \psi_0}{\partial \xi\, \partial \eta} = \frac{1}{4c_{44}}\frac{\partial H}{\partial x}, \quad \frac{\partial G_0}{\partial \xi\, \partial \eta} = \frac{1}{4c_{44}}\frac{\partial H}{\partial y}. \tag{A7}$$

Now define

$$H_0(x,y,z,t) = H_0\left(x, y, \frac{\xi+\eta}{2}, \frac{\eta-\xi}{2c_2}\right) = \frac{1}{4c_{44}}\int\left(\int H\left(x, y, \frac{\xi+\eta}{2}, \frac{\eta-\xi}{2c_2}\right)\mathrm{d}\xi\right)\mathrm{d}\eta \tag{A8}$$

of which the right-hand side means that the integration of H for η takes place once the integration for ξ has taken place.

Clearly, if we let

$$\psi_0 = \frac{\partial H_0}{\partial x}, \quad G_0 = \frac{\partial H_0}{\partial y}, \tag{A9}$$

then eqn (A7) is satisfied; and since function H satisfies eqn (17), the functions ψ_0 and G_0 represented by eqns (A9) satisfy eqn (A1).

APPENDIX B

In this section we prove a generalized Almansi's theorem, stated as follows:

Let R be a region of the (x, y, z)-space such that a straight line parallel to the z-axis intersects the boundary of R at no more than two points. Let $F_n(x, y, z)$ be a solution of

$$\prod_{i=1}^{n} \nabla_i^2 F_n = \nabla_1^2 \nabla_2^2 \ldots \nabla_{n-1}^2 \nabla_n^2 F_n = 0 \quad \text{in } R, \tag{B1}$$

where

$$\nabla_i^2 = \Lambda + c_i^2 D^2, \quad D = \frac{\partial}{\partial z}, \tag{B2}$$

and the c_i are constants. Then F_n admits the representation

$$F_n(x, y, z) = F_{n-1}(x, y, z) + z^m F^{(n)}(x, y, z), \tag{B3}$$

where F_{n-1} and $F^{(n)}$, respectively, satisfy

$$\prod_{i=1}^{n-1} \nabla_i^2 F_{n-1} = 0, \tag{B4}$$

$$\nabla_n^2 F^{(n)} = 0, \tag{B5}$$

and $m(0 \leqslant m \leqslant n-1)$ is the number of the coefficients c_i^2 $(i = 1, 2, \ldots, n-1)$ which are equal to c_n^2.
To prove this theorem, we need some preliminaries:

(1) We first note the trivial identities

$$\nabla_i^2[A(z)B(x, y, z)] = A\nabla_i^2 B + c_i^2(A_{,zz}B + 2A_{,z}B_{,z}), \tag{B6}$$

$$\nabla_j^2 F^{(i)} = (c_j^2 - c_i^2)F^{(i)}_{,zz} = (c_j^2 - c_i^2)D^2 F^{(i)} \quad \text{if } \nabla_i^2 F^{(i)} = 0, \tag{B7}$$

and observe that $F_{,z}$ also satisfies eqn (B7).
(2) As a further preliminary, we prove that if $F(x, y, z)$ satisfies $\nabla_i^2 F = 0$ and $m > 0$ is an integer, then

$$\nabla_i^{2m}(z^k F) = \begin{cases} 0 & \text{for} \quad k = m-1 \\ 2^m m!\, c_i^{2m} D^m F & \text{for} \quad k = m. \end{cases} \tag{B8}$$

Let $k = m-1$. Clearly, eqn (B8) holds for $m = 1$ and 2. We proceed by induction. Thus, assume

$$\nabla_i^{2(m-1)}(z^{m-2}F) = 0.$$

Then, by hypothesis and eqn (B6),

$$\begin{aligned} \nabla_i^{2m}(z^{m-1}F) &= \nabla_i^{2(m-1)}\nabla_i^2(z^{m-1}F) \\ &= (m-1)(m-2)c_i^2\nabla_i^{2(m-1)}(z^{m-3}F) + 2(m-1)c_i^2\nabla_i^{2(m-1)}(z^{m-2}F_{,z}) = 0. \end{aligned}$$

Next, let $k = m$. Again eqn (B8) evidently holds for $m = 1$, in fact, when $m = k = 1$, $\nabla_i^2(zF) = 2c_i^2 F_{,z} = 2c_i^2 DF$. Using induction once more, assume

$$\nabla_i^{2(m-1)}(z^{m-1}F) = 2^{m-1}(m-1)!\,c_i^{2(m-1)}D^{(m-1)}F.$$

Then,

$$\begin{aligned} \nabla_i^{2m}(z^m F) &= \nabla_i^{2(m-1)}\nabla_i^2(z^m F) \\ &= mc_i^2[(m-1)\nabla_i^{2(m-1)}(z^{m-2}F) + 2\nabla_i^{2(m-1)}(z^{m-1}F_{,z})] \\ &= 2^m m!\,c_i^{2m}D^m F. \end{aligned}$$

This completes the proof of eqn (B8).
Turning to the proof of the theorem, we may assume with no loss of generality that

$$\begin{aligned} &c_i^2 = c_n^2 \quad (i = n-m, n-m+1, \ldots, n-1) \\ &c_i^2 \neq c_n^2 \quad (i = 1, 2, \ldots, n-m-1). \end{aligned} \tag{B9}$$

In view of eqn (B3), F_{n-1} admits the representation

$$F_{n-1} = F_n - z^m F^{(n)}, \tag{B10}$$

and eqn (B4), by virtue of eqn (B10), becomes

$$\prod_{i=1}^{n-1} \nabla_i^2 F_{n-1} = \prod_{i=1}^{n-1} \nabla_i^2 F_n - \prod_{i=1}^{n-1} \nabla_i^2 (z^m F^{(n)}) = 0. \tag{B11}$$

Assume that $F^{(n)}$ satisfies eqn (B5). Then, by virtue of eqns (B7) and (B8),

$$\prod_{i=1}^{n-1} \nabla_i^2 (z^m F^{(n)}) = \prod_{i=1}^{n-m-1} \nabla_i^2 [\nabla_n^{2m} (z^m F^{(n)})]$$

$$= 2^m m\,!\, c_n^{2m} D^m \prod_{i=1}^{n-m-1} \nabla_i^2 F^{(n)} = H D^{2n-m-2} F^{(n)}, \tag{B12}$$

where

$$H = 2^m m\,! c_n^{2m} \prod_{i=1}^{n-m-1} (c_i^2 - c_n^2). \tag{B13}$$

In view of eqn (B12), eqn (B11) becomes

$$H D^{2n-m-2} F^{(n)} = \prod_{i=1}^{n-1} \nabla_i^2 F_n. \tag{B14}$$

Equation (B14) evidently has a particular solution

$$F_*^{(n)} = \frac{1}{H} D^{-2n+m+2} \prod_{i=1}^{n-1} \nabla_i^2 F_n, \tag{B15}$$

where the operator D^{-1} is defined as

$$D^{-1} G(x, y, z) = \int_{z_0}^{z} G(x, y, \xi)\, \mathrm{d}\xi, \tag{B16}$$

and z_0 is a point on the boundary of R. On the other hand, the solution for the homogeneous equation of (B14), which is of the form

$$D^{2n-m-2} F^{(n)} = 0, \tag{B17}$$

may takes the form

$$F_0^{(n)} = \sum_{k=0}^{2n-m-3} z^k f_k(x, y), \tag{B18}$$

where $f_k(x, y)$ are arbitrary functions. Thus, we may write the general solution of eqn (B14) as

$$F^{(n)} = F_0^{(n)} + F_*^{(n)}. \tag{B19}$$

Substituting eqn (B19) into eqn (B5), we have

$$\nabla_n^2 F_0^{(n)} + \nabla_n^2 F_*{}^{(n)} = 0. \tag{B20}$$

And by virtue of eqns (B18), (B15) and (B1), we have

$$\nabla_n^2 F_0^{(n)} = \nabla_n^2 \sum_{k=0}^{2n-m-3} z^k f_k(x, y)$$

$$= z^{2n-m-3} \Lambda f_{2n-m-3} + z^{2n-m-4} \Lambda f_{2n-m-4}$$

$$+ \sum_{k=0}^{2n-m-5} [\Lambda f_k + (k+2)(k+1) c_n^2 f_{k+2}] z^k, \tag{B21}$$

$$D^{2n-m-2} \nabla_n^2 F_*^{(n)} = 0. \tag{B22}$$

In view of eqn (B22), we know $\nabla_n^2 F_*^{(n)}$ takes the following form

$$\nabla_n^2 F_*^{(n)} = \sum_{k=0}^{2n-m-3} z^k h_k(x, y), \tag{B23}$$

where $h_k(x, y)$ are all known since F_n is known and $F_*^{(n)}$ is determined by eqn (B15).

Substituting eqns (B21) and (B23) into eqn (B20) and comparing the coefficients before z^k, we have

$$\Lambda f_{2n-m-3}+h_{2n-m-3}=0 \tag{B24}$$

$$\Lambda f_{2n-m-4}+h_{2n-m-4}=0 \tag{B25}$$

$$\Lambda_k f_k+(k+2)(k+1)c_n^2 f_{k+2}+h_k=0, \quad (k=0,1,\ldots,2n-m-5). \tag{B26}$$

Each of the eqns (B24)–(B26) is Poisson equation which is very easy to find a particular solution. That means f_k may be obtained one by one. Thus, by the construction of f_k, the $F^{(n)}$ obtained from eqn (B19) satisfies eqn (B5). Meanwhile, the F_{n-1} obtained by substituting this $F^{(n)}$ into eqn (B10) also satisfies eqn (B4). This completes the proof.

APPENDIX C

$$n_{61}=c_{44}[-(m_{13}m_{42})+m_{12}m_{43}]s_0s_1(-m_1+m_2s_1^2)$$

$$n_{62}=-\{c_{44}[-(m_{13}m_{41})+m_{11}m_{43}]s_0s_2(-m_1+m_2s_2^2)\}$$

$$n_{63}=c_{44}[-(m_{12}m_{41})+m_{11}m_{42}]s_0s_3(-m_1+m_2s_3^2)$$

$$n_{60}=-(m_{13}m_{32}m_{41})+m_{12}m_{33}m_{41}+m_{13}m_{31}m_{42}-m_{11}m_{33}m_{42}-m_{12}m_{31}m_{43}+m_{11}m_{32}m_{43}$$

$$n_{51}=c_{44}[-(m_{13}m_{42})+m_{12}m_{43}]s_0s_1(-m_1+m_2s_1^2)$$

$$n_{52}=-\{c_{44}[-(m_{13}m_{41})+m_{11}m_{43}]s_0s_2(-m_1+m_2s_2^2)\}$$

$$n_{53}=c_{44}[-(m_{12}m_{41})+m_{11}m_{42}]s_0s_3(-m_1+m_2s_3^2)$$

$$n_{50}=m_{13}m_{32}m_{41}-m_{12}m_{33}m_{41}-m_{13}m_{31}m_{42}+m_{11}m_{33}m_{42}+m_{12}m_{31}m_{43}-m_{11}m_{32}m_{43}$$

$$n_{71}=c_{44}[-(m_{13}m_{42})+m_{12}m_{43}]s_0[-(m_3s_1^2)+c_{11}\varepsilon_{11}+c_{44}s_1^4\varepsilon_{33}]$$

$$n_{72}=-\{c_{44}[-(m_{13}m_{41})+m_{11}m_{43}]s_0[-(m_3s_2^2)+c_{11}\varepsilon_{11}+c_{44}s_2^4\varepsilon_{33}]\}$$

$$n_{73}=c_{44}[-(m_{12}m_{41})+m_{11}m_{42}]s_0[-(m_3s_3^2)+c_{11}\varepsilon_{11}+c_{44}s_3^4\varepsilon_{33}]$$

$$n_{81}=c_{44}[-(m_{13}m_{42})+m_{12}m_{43}]s_0(c_{11}e_{15}-m_4s_1^2+c_{44}e_{33}s_1^4)$$

$$n_{82}=-\{c_{44}[-(m_{13}m_{41})+m_{11}m_{43}]s_0(c_{11}e_{15}-m_4s_2^2+c_{44}e_{33}s_2^4)\}$$

$$n_{83}=c_{44}[-(m_{12}m_{41})+m_{11}m_{42}]s_0(c_{11}e_{15}-m_4s_3^2+c_{44}e_{33}s_3^4)$$

$$l_{81}=(m_{33}m_{42}-m_{32}m_{43})(c_{11}e_{15}-m_4s_1^2+c_{44}e_{33}s_1^4)$$

$$l_{82}=[-(m_{33}m_{41})+m_{31}m_{43}](c_{11}e_{15}-m_4s_2^2+c_{44}e_{33}s_2^4)$$

$$l_{83}=(m_{32}m_{41}-m_{31}m_{42})(c_{11}e_{15}-m_4s_3^2+c_{44}e_{33}s_3^4)$$

$$l_{71}=(m_{33}m_{42}-m_{32}m_{43})[-(m_3s_1^2)+c_{11}\varepsilon_{11}+c_{44}s_1^4\varepsilon_{33}]$$

$$l_{72}=[-(m_{33}m_{41})+m_{31}m_{43}][-(m_3s_2^2)+c_{11}\varepsilon_{11}+c_{44}s_2^4\varepsilon_{33}]$$

$$l_{73}=(m_{32}m_{41}-m_{31}m_{42})[-(m_3s_3^2)+c_{11}\varepsilon_{11}+c_{44}s_3^4\varepsilon_{33}]$$

$$l_{51}=(m_{33}m_{42}-m_{32}m_{43})s_1(-m_1+m_2s_1^2)$$

$$l_{52}=[-(m_{33}m_{41})+m_{31}m_{43}]s_2(-m_1+m_2s_2^2)$$

$$l_{53}=[(m_{32}m_{41}-m_{31}m_{42})s_3(-m_1+m_2s_3^2)$$

$$l_{61}=(m_{33}m_{42}-m_{32}m_{43})s_1(-m_1+m_2s_1^2)$$

$$l_{62}=[-(m_{33}m_{41})+m_{31}m_{43}]s_2(-m_1+m_2s_2^2)$$

$$l_{63}=(m_{32}m_{41}-m_{31}m_{42})s_3(-m_1+m_2s_3^2)$$

$$b_{81}=[-(m_{13}m_{32})+m_{12}m_{33}](c_{11}e_{15}-m_4s_1^2+c_{44}e_{33}s_1^4)$$

$$b_{82}=(m_{13}m_{31}-m_{11}m_{33})(c_{11}e_{15}-m_4s_2^2+c_{44}e_{33}s_2^4)$$

$$b_{83}=[-(m_{12}m_{31})+m_{11}m_{32}](c_{11}e_{15}-m_4s_3^2+c_{44}e_{33}s_3^4)$$

$$b_{71}=[-(m_{13}m_{32})+m_{12}m_{33}][-(m_3s_1^2)+c_{11}\varepsilon_{11}+c_{44}s_1^4\varepsilon_{33}]$$

$$b_{72}=(m_{13}m_{31}-m_{11}m_{33})[-(m_3s_2^2)+c_{11}\varepsilon_{11}+c_{44}s_2^4\varepsilon_{33}]$$

$$b_{73}=[-(m_{12}m_{31})+m_{11}m_{32}][-(m_3s_3^2)+c_{11}\varepsilon_{11}+c_{44}s_3^4\varepsilon_{33}]$$

$$b_{51} = [-(m_{13}m_{32})+m_{12}m_{33}]s_1(-m_1+m_2s_1^2)$$

$$b_{52} = (m_{13}m_{31}-m_{11}m_{33})s_2(-m_1+m_2s_2^2)$$

$$b_{53} = [-(m_{12}m_{31})+m_{11}m_{32}]s_3(-m_1+m_2s_3^2)$$

$$b_{61} = [-(m_{13}m_{32})+m_{12}m_{33}]s_1(-m_1+m_2s_1^2)$$

$$b_{62} = (m_{13}m_{31}-m_{11}m_{33})s_2(-m_1+m_2s_2^2)$$

$$b_{63} = [-(m_{12}m_{31})+m_{11}m_{32}]s_3(-m_1+m_2s_3^2)$$

$$d_z = 2\pi[-(m_{13}m_{32}m_{41})+m_{12}m_{33}m_{41}+m_{13}m_{31}m_{42}-m_{11}m_{33}m_{42}-m_{12}m_{31}m_{43}+m_{11}m_{32}m_{43}]$$

$$d_x = 2\pi c_{44}(m_{13}m_{32}m_{41}-m_{12}m_{33}m_{41}-m_{13}m_{31}m_{42}+m_{11}m_{33}m_{42}+m_{12}m_{31}m_{43}-m_{11}m_{32}m_{43})s_0$$

$$d_d = 2\pi(m_{13}m_{32}m_{41}-m_{12}m_{33}m_{41}-m_{13}m_{31}m_{42}+m_{11}m_{33}m_{42}+m_{12}m_{31}m_{43}-m_{11}m_{32}m_{43})$$

$$\begin{aligned} m_{11} = {} & c_{13}e_{15}^2s_1+c_{13}e_{15}e_{31}s_1-c_{11}e_{15}e_{33}s_1+c_{33}e_{15}^2s_1^3+2c_{33}e_{15}e_{31}s_1^3 \\ & +c_{33}e_{31}^2s_1^3-2c_{13}e_{15}e_{33}s_1^3-2c_{13}e_{31}e_{33}s_1^3-c_{44}e_{31}e_{33}s_1^3+c_{11}e_{33}^2s_1^3 \\ & -c_{44}e_{33}^2s_1^5+c_{13}^2s_1\varepsilon_{11}-c_{11}c_{33}s_1\varepsilon_{11}+c_{13}c_{44}s_1\varepsilon_{11}+c_{33}c_{44}s_1^3\varepsilon_{11} \\ & -c_{13}^2s_1^3\varepsilon_{33}+c_{11}c_{33}s_1^3\varepsilon_{33}-c_{13}c_{44}s_1^3\varepsilon_{33}-c_{33}c_{44}s_1^5\varepsilon_{33} \end{aligned}$$

$$\begin{aligned} m_{12} = {} & c_{13}e_{15}^2s_2+c_{13}e_{15}e_{31}s_2-c_{11}e_{15}e_{33}s_2+c_{33}e_{15}^2s_2^3+2c_{33}e_{15}e_{31}s_2^3 \\ & +c_{33}e_{31}^2s_2^3-2c_{13}e_{15}e_{33}s_2^3-2c_{13}e_{31}e_{33}s_2^3-c_{44}e_{31}e_{33}s_2^3+c_{11}e_{33}^2s_2^3 \\ & -c_{44}e_{33}^2s_2^5+c_{13}^2s_2\varepsilon_{11}-c_{11}c_{33}s_2\varepsilon_{11}+c_{13}c_{44}s_2\varepsilon_{11}+c_{33}c_{44}s_2^3\varepsilon_{11} \\ & -c_{13}^2s_2^3\varepsilon_{33}+c_{11}c_{33}s_2^3\varepsilon_{33}-c_{13}c_{44}s_2^3\varepsilon_{33}-c_{33}c_{44}s_2^5\varepsilon_{33} \end{aligned}$$

$$\begin{aligned} m_{13} = {} & c_{13}e_{15}^2s_3+c_{13}e_{15}e_{31}s_3-c_{11}e_{15}e_{33}s_3+c_{33}e_{15}^2s_3^3+2c_{33}e_{15}e_{31}s_3^3 \\ & +c_{33}e_{31}^2s_3^3-2c_{13}e_{15}e_{33}s_3^3-2c_{13}e_{31}e_{33}s_3^3-c_{44}e_{31}e_{33}s_3^3+c_{11}e_{33}^2s_3^3 \\ & -c_{44}e_{33}^2s_3^5+c_{13}^2s_3\varepsilon_{11}-c_{11}c_{33}s_3\varepsilon_{11}+c_{13}c_{44}s_3\varepsilon_{11}+c_{33}c_{44}s_3^3\varepsilon_{11} \\ & -c_{13}^2s_3^3\varepsilon_{33}+c_{11}c_{33}s_3^3\varepsilon_{33}-c_{13}c_{44}s_3^3\varepsilon_{33}-c_{33}c_{44}s_3^5\varepsilon_{33} \end{aligned}$$

$$\begin{aligned} m_{21} = {} & -(c_{11}e_{15}^2)-c_{13}e_{15}^2s_1^2-c_{13}e_{15}e_{31}s_1^2+c_{44}e_{31}^2s_1^2+c_{11}e_{15}e_{33}s_1^2 \\ & +c_{44}e_{31}e_{33}s_1^4-c_{11}c_{44}\varepsilon_{11}-c_{13}c_{44}s_1^2\varepsilon_{11}+c_{11}c_{44}s_1^2\varepsilon_{33}+c_{13}c_{44}s_1^4\varepsilon_{33} \end{aligned}$$

$$\begin{aligned} m_{22} = {} & -(c_{11}e_{15}^2)-c_{13}e_{15}^2s_2^2-c_{13}e_{15}e_{31}s_2^2+c_{44}e_{31}^2s_2^2+c_{11}e_{15}e_{33}s_2^2 \\ & +c_{44}e_{31}e_{33}s_2^4-c_{11}c_{44}\varepsilon_{11}-c_{13}c_{44}s_2^2\varepsilon_{11}+c_{11}c_{44}s_2^2\varepsilon_{33}+c_{13}c_{44}s_2^4\varepsilon_{33} \end{aligned}$$

$$\begin{aligned} m_{23} = {} & -(c_{11}e_{15}^2)-c_{13}e_{15}^2s_3^2-c_{13}e_{15}e_{31}s_3^2+c_{44}e_{31}^2s_3^2+c_{11}e_{15}e_{33}s_3^2 \\ & +c_{44}e_{31}e_{33}s_3^4-c_{11}c_{44}\varepsilon_{11}-c_{13}c_{44}s_3^2\varepsilon_{11}+c_{11}c_{44}s_3^2\varepsilon_{33}+c_{13}c_{44}s_3^4\varepsilon_{33} \end{aligned}$$

$$\begin{aligned} m_{31} = {} & -(c_{11}e_{15}^2)-c_{13}e_{15}^2s_1^2-c_{13}e_{15}e_{31}s_1^2+c_{44}e_{31}^2s_1^2+c_{11}e_{15}e_{33}s_1^2 \\ & +c_{44}e_{31}e_{33}s_1^4-c_{11}c_{44}\varepsilon_{11}-c_{13}c_{44}s_1^2\varepsilon_{11}+c_{11}c_{44}s_1^2\varepsilon_{33}+c_{13}c_{44}s_1^4\varepsilon_{33} \end{aligned}$$

$$\begin{aligned} m_{32} = {} & -(c_{11}e_{15}^2)-c_{13}e_{15}^2s_2^2-c_{13}e_{15}e_{31}s_2^2+c_{44}e_{31}^2s_2^2+c_{11}e_{15}e_{33}s_2^2 \\ & +c_{44}e_{31}e_{33}s_2^4-c_{11}c_{44}\varepsilon_{11}-c_{13}c_{44}s_2^2\varepsilon_{11}+c_{11}c_{44}s_2^2\varepsilon_{33}+c_{13}c_{44}s_2^4\varepsilon_{33} \end{aligned}$$

$$\begin{aligned} m_{33} = {} & -(c_{11}e_{15}^2)-c_{13}e_{15}^2s_3^2-c_{13}e_{15}e_{31}s_3^2+c_{44}e_{31}^2s_3^2+c_{11}e_{15}e_{33}s_3^2 \\ & +c_{44}e_{31}e_{33}s_3^4-c_{11}c_{44}\varepsilon_{11}-c_{13}c_{44}s_3^2\varepsilon_{11}+c_{11}c_{44}s_3^2\varepsilon_{33}+c_{13}c_{44}s_3^4\varepsilon_{33} \end{aligned}$$

$$\begin{aligned} m_{41} = {} & e_{15}^2e_{31}s_1+e_{15}e_{31}^2s_1+e_{15}^2e_{33}s_1^3+e_{15}e_{31}e_{33}s_1^3+c_{13}e_{31}s_1\varepsilon_{11} \\ & +c_{44}e_{31}s_1\varepsilon_{11}-c_{11}e_{33}s_1\varepsilon_{11}+c_{44}e_{33}s_1^3\varepsilon_{11}+c_{11}e_{15}s_1\varepsilon_{33}+c_{13}e_{15}s_1^3\varepsilon_{33} \end{aligned}$$

$$\begin{aligned} m_{42} = {} & e_{15}^2e_{31}s_2+e_{15}e_{31}^2s_2+e_{15}^2e_{33}s_2^3+e_{15}e_{31}e_{33}s_2^3+c_{13}e_{31}s_2\varepsilon_{11} \\ & +c_{44}e_{31}s_2\varepsilon_{11}-c_{11}e_{33}s_2\varepsilon_{11}+c_{44}e_{33}s_2^3\varepsilon_{11}+c_{11}e_{15}s_2\varepsilon_{33}+c_{13}e_{15}s_2^3\varepsilon_{33} \end{aligned}$$

$$\begin{aligned} m_{43} = {} & e_{15}^2e_{31}s_3+e_{15}e_{31}^2s_3+e_{15}^2e_{33}s_3^3+e_{15}e_{31}e_{33}s_3^3+c_{13}e_{31}s_3\varepsilon_{11} \\ & +c_{44}e_{31}s_3\varepsilon_{11}-c_{11}e_{33}s_3\varepsilon_{11}+c_{44}e_{33}s_3^3\varepsilon_{11}+c_{11}e_{15}s_3\varepsilon_{33}+c_{13}e_{15}s_3^3\varepsilon_{33}. \end{aligned}$$